Reheating and turbulence

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We show that the "turbulent" particle spectra found in numerical simulations of the behavior of matter fields during reheating admit a simple interpretation in terms of hydrodynamic models of the reheating period. We predict a particle number spectrum $n_k \propto k^{-\alpha}$ with $\alpha \sim 2$ for $k \to 0$.

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I. INTRODUCTION

The reheating period in the early Universe stands out as a challenge to theorists due to the close interrelationship of nonlinear and gravitational phenomena in its unfolding (see Refs. [1–5]). The observation that, due to the high occupation numbers produced during preheating, most of the physics of reheating may be understood in terms of the behavior of nonlinear classical waves [1,6] has been the key to substantial progress. The authors of [1–5] have undertaken systematic numerical simulations of the behavior of matter fields during reheating, finding that the reheating period is actually composed of three consecutive phases: an early one or preheating, where the dominant effect is the parametric amplification of matter fields out of the dynamical inflaton and gravitational backgrounds [7], an intermediate stage where the dominant phenomenon is the redistribution of energy among matter field modes through rescattering (in the terminology of [1–5]), and a final stage where thermal equilibrium sets in. During the intermediate stage, spectra of occupation numbers for the matter fields reduce to simple power laws both in the infrared and ultraviolet limits. As noted in [1–5], this behavior suggests a connection between the physics of reheating and the phenomena of weak turbulence [8], but to the best of our knowledge no theoretical prediction for the exponents involved is available. Our goal is to provide these theoretical estimates.

In this paper we shall follow this same trend of ideas, by observing that, from the macroscopic point of view, a stochastic ensemble of classical waves may be described by a conserved energy momentum tensor subject to the second law of thermodynamics. There is therefore an equivalent fluid description, consisting of a fluid whose energy momentum tensor and equation of state reproduce the observed ones for the microscopic fluctuations. Solving the dynamics of this equivalent fluid yields answers to all relevant questions concerning the behavior of the actual fluctuations.

An immediate consequence of energy momentum conservation and the second law is that, when velocities are low, the phenomenological fluid may be described within the Eckart theory of dissipative fluids [9] (for an analysis of the limitations of Eckart's theory see [10]). It follows that it obeys a continuity equation and a curved space-time Navier-Stokes one. The "turbulent" spectra found in numerical simulations correspond to the self-similar solutions discussed long ago by Chandrasekhar [11]. They appear in the discussion of decaying turbulence (which is the case relevant to cosmology), as opposed to turbulence driven by some external means. The Chandrasekhar solutions are built on the Heisenberg closure hypothesis [12] (see [13–15] for a general discussion of turbulence). They were generalized to Friedmann-Robertson-Walker (FRW) backgrounds by Tomita *et al.* [16]. These solutions agree with the Kolmogorov 1941 theory in the inertial range [15], failing to reproduce observations for very small eddies. Fortunately, we are most interested in the opposite limit of very large eddies, where it is trustworthy (we wish to point out that the applicability of Kolmogorov's spectrum to large scale turbulence should not be taken for granted [17]). With minor adjustments, Tomita's analysis of turbulence decay in FRW space-times also provides a solution to the evolution of our equivalent fluid.

The rest of the paper is organized as follows. In the next section we provide a brief summary of hydrodynamics in flat and expanding universes, in order to set up the language for the rest of the paper, and introduce the self-similar solutions. In Sec. III we proceed to discuss the equivalent fluid description of field fluctuations, and how to extract the particle spectrum therefrom. In Sec. IV we place the self-similar solutions in the context of reheating. We state our main conclusions in the final section. We provide a rough estimate of the shear viscosity during reheating in the Appendix.

II. HYDRODYNAMIC FLOWS

A. Flows in flat space-time

The equations governing the dynamics of a fluid in local thermodynamic equilibrium are the continuity and Navier-Stokes ones, which, in the case of flat space-time, read

$$\frac{\partial \rho}{\partial t} + (\mathbf{U} \cdot \boldsymbol{\nabla})\rho = 0 \tag{1}$$

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \boldsymbol{\nabla})\mathbf{U} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{U} \tag{2}$$

where we have assumed incompressibility, valid when typical velocities are much smaller than the sound velocity; $\nu = \eta/\rho$ is the kinematic shear viscosity. The transition from laminar to turbulent motion can be universally described by the dimensionless ‘‘Reynolds’’ number:

$$R=\frac{UL}{\nu} \tag{3}$$

where U is a typical velocity and L a typical length scale. This number represents the order of magnitude of the ratio of the inertial to the viscous term. Low Reynolds numbers correspond to laminar motion, while high ones suggest turbulent behavior.

In general, the velocity profile displays variations in space and time. This implies that the flow must be described probabilistically. Thus, each quantity involved in Eqs. (1),(2) is divided in its mean value and a fluctuation from it; for example, we write $\mathbf{U}=\overline{U}+u$, where u stands for the fluctuating part of the velocity. In the case where motion is isotropic, the mean value $\overline{U}$ for the velocity must be zero, since otherwise there would be a preferred direction.

To analyze the system’s behavior, we define the two-point one-time correlation function for the velocity:

$$R_{ij}(\mathbf{x},\mathbf{x}',t)=\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle. \tag{4}$$

In the case of homogeneous and isotropic motion, this correlation must be only a function of the time t and the distance between $\mathbf{x}$ and $\mathbf{x}'$, i.e. $R_{ij}(\mathbf{x},\mathbf{x}',t)=R_{ij}(r,t)$, where $r=|\mathbf{x}-\mathbf{x}'|$. Observe that $R_{ii}(0,t)$ (summation over repeated indices must be understood) is twice the average energy density of the flow at time t. From Eq. (2) we obtain the equation that this correlation must obey: namely,

$$\frac{\partial}{\partial t}R_{ij}(r,t)=T_{ij}(r,t)+P_{ij}(r,t)+2\nu\nabla^2R_{ij}(r,t) \tag{5}$$

where

$$P_{ij}(r,t)=\frac{1}{\rho}\left(\frac{\partial}{\partial r_i}\langle p(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle -\frac{\partial}{\partial r_j}\langle p(\mathbf{x}',t)u_i(\mathbf{x},t)\rangle\right) \tag{6}$$

and

$$T_{ij}(r,t)=\frac{\partial}{\partial r_k}\langle u_i(\mathbf{x},t)u_k(\mathbf{x},t)u_j(\mathbf{x}',t) -u_i(\mathbf{x},t)u_k(\mathbf{x}',t)u_j(\mathbf{x}',t)\rangle. \tag{7}$$

The tensor T_{ij} comes form the inertia term in the Navier-Stokes equation and, as it involves a product of third order in the velocity, reflects the fact that there is not a close set of equations for the correlations of successive orders but there is a hierarchy of equations instead. The problem of closing that hierarchy is known as the ‘‘moment closure problem’’ [18]. Let us call $\Phi_{ij}(k,t)$ the Fourier transform of $R_{ij}(r,t)$. Then the energy density becomes

$$\frac{1}{2}R_{ii}(0,t)=\int E(k,t)dk,$$

where

$$E(k,t)=\frac{1}{2}\int \Phi_{ii}(\mathbf{k},t)k^2d\Omega(\mathbf{k}) \tag{8}$$

is the energy density stored in eddies of size k^{-1}. Defining Γ_{ij} as the Fourier transform of T_{ij}, we obtain from Eq. (5) the equation of balance of the energy spectrum:

$$-\frac{\partial}{\partial t}E(k,t)=T(k,t)+2\nu k^2E(k,t) \tag{9}$$

where

$$T(k,t)=-\frac{1}{2}\int \Gamma_{ii}(\mathbf{k},t)k^2d\Omega(\mathbf{k}). \tag{10}$$

The inertia term $T(k,t)$ is the one that contains the mode-mode interaction, and its effect is to drain energy from the more energetic modes—typically the bigger ones—to the ones where there is major viscous dissipation—the smaller ones.

B. Flows in expanding universes

For a curved space-time, in particular a Friedmann-Robertson-Walker universe with zero spatial curvature $[ds^2=-dt^2+a^2(t)(dx^2+dy^2+dz^2)]$, the generalization of the above arguments has been considered by many authors [19–23]. We follow the analysis of Tomita *et al.* [16], in which they obtain the solution for the energy spectrum in the case of homogeneous, isotropic and incompressible turbulence.

In a generic space-time, we describe fluid flow from the energy density ρ, pressure p and four-velocity U. The symmetries of the FRW solution suggest using instead the comoving three-velocity $u^i=U^i/U^0$; if $U^i\ll U^0$ the flow is nonrelativistic, and if $\nabla\mathbf{u}=0$, it is incompressible $[\mathbf{u}=(u^1,u^2,u^3)]$. Later on, we shall also use the physical three-velocity $v=a(t)u$.

The corresponding continuity and Navier-Stokes equations for a Robertson-Walker background are obtained by the condition of conservation of the energy-momentum tensor [9]. For a nonrelativistic incompressible fluid, with shear viscosity $\eta=\nu(p+\rho)$ (but no bulk viscosity), these reduce to

$$\frac{\partial\rho}{\partial t}+3\frac{\dot{a}}{a}(p+\rho)=0 \tag{11}$$

$$\frac{\partial\mathbf{u}}{\partial t}+\left[(\mathbf{u}\cdot\nabla)+\frac{\partial\ln[(p+\rho)a^5]}{\partial t}\right]\mathbf{u} =-\frac{\boldsymbol{\nabla}p}{a^2(p+\rho)}+\frac{1}{a^2}\nu\boldsymbol{\nabla}^2\mathbf{u} \tag{12}$$

where we have assumed that $p+\rho$ depends only on time. For the physical three-velocity $\mathbf{v}$, the corresponding Navier-Stokes equation reads

$$\frac{\partial \mathbf{v}}{\partial t}+\left[\frac{1}{a}(\mathbf{v}\cdot\boldsymbol{\nabla})+\frac{\partial \ln[(p+\rho)a^4]}{\partial t}\right]\mathbf{v} = -\frac{\boldsymbol{\nabla} p}{a(p+\rho)}+\frac{1}{a^2}\nu\boldsymbol{\nabla}^2\mathbf{v}. \qquad (13)$$

In obtaining Eqs. (11)–(13) we have neglected possible perturbations to the FRW metric. The corresponding equations considering fluctuations in the metric ($g_{\mu\nu}=g^0_{\mu\nu}+h_{\mu\nu}$) have been obtained by Weinberg [9]. The continuity equation is not corrected by gravitational perturbations, while in the Navier-Stokes equation the metric fluctuations appear explicitly only within the shear viscosity term. It can be demonstrated that these terms involving metric fluctuations are negligible for scales that are inside the horizon [24]. For scales bigger than the Hubble radius, since dissipation through viscosity is not effective anyway, we may still use the unperturbed Navier-Stokes equation.

The operation of Fourier transforming in the case of a Robertson-Walker cosmology is done in terms of comoving wave-numbers. In doing so, the following equation for the energy spectrum is obtained:

$$-\frac{\partial}{\partial t}E(k,t)=T(k,t)+2\left\{\frac{\nu k^2}{a^2}+\frac{\partial \ln[(p+\rho)a^4]}{\partial t}\right\}E(k,t) \qquad (14)$$

where the relationship between $E(k,t)$ and $\Phi_{ij}(k,t)$ as well as between $T(k,t)$ and $\Gamma_{ij}(k,t)$ is the same as that for a flat space-time, if we define R_{ij} and T_{ij} from correlations of physical quantities, as follows:

$$R_{ij}(r,t)=a^2\langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)\rangle \qquad (15)$$

$$T_{ij}(r,t)=a^2\frac{\partial}{\partial r_k}[\langle u_i(\mathbf{x},t)u_k(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)\rangle - \langle u_i(\mathbf{x},t)u_k(\mathbf{x}+\mathbf{r},t)u_j(\mathbf{x}+\mathbf{r},t)\rangle]. \qquad (16)$$

C. Self-similar flows in flat and expanding universes

As we have seen in the previous section, the key element in the description of the flow is the energy spectrum $E(k)$ [Eq. (8)], which is the solution of the balance equation [Eq. (9)]. In it, the right hand side contains the viscous dissipation as well as the inertial force $T(k,t)$. The overall effect of this term is to transfer energy from a given scale to smaller ones through mode-mode coupling; thus it is natural to model the action of the inertia term as a source of viscous dissipation, where the effective turbulent viscosity for a given mode depends on the motion of all smaller eddies [14]. By providing closure, that is, writing this effective viscosity in terms of the spectrum itself, a closed evolution equation for $E(k)$ is obtained. Concretely, Heisenberg [12] proposed the ansatz

$$\int_0^k T(k',t)dk'=2\nu(k,t)\int_0^k E(k',t)k'^2 dk' \qquad (17)$$

where

$$\nu(k,t)=A_{flat}\int_k^\infty\sqrt{\frac{E(k',t)}{k'^3}}dk' \qquad (18)$$

and A_{flat} is a dimensionless constant. With this hypothesis (known as the Heisenberg hypothesis) as the solution to the closure problem, Chandrasekhar [11] has obtained the energy spectrum for decaying turbulence, assuming that there is a stage in the decay where the bigger eddies have a sufficient amount of energy to maintain an equilibrium distribution, thus requiring that the solution for the spectrum should be self similar. With this consideration into account he obtained an energy spectrum:

$$E(k,t)=\frac{1}{A^2_{flat}k_0^3t_0^2}\sqrt{\frac{t_0}{t}}F\left(\frac{k\sqrt{t}}{k_0\sqrt{t_0}}\right) \qquad (19)$$

where k_0 and t_0 are initial conditions (namely, the wave number corresponding to the bigger eddy and its typical time of evolution). The function F obeys the equation

$$\frac{1}{4}\int_0^x\left[F(x')-x'\frac{dF(x')}{dx'}\right]dx' = \left\{\nu k_0^2t_0+\int_x^\infty\frac{\sqrt{F(x')}}{x'^{3/2}}dx'\right\}\int_0^x F(x')x'^2dx' \qquad (20)$$

which predicts a Kolmogorov type behavior for an inviscid fluid $[R\to\infty, R=(\nu k_0^2 t_0)^{-1}]$ in the ultraviolet limit:

$$F(x)\to \text{const}\times x^{-5/3} \quad (\nu=0, x\to\infty), \qquad (21)$$

while, for nonzero viscosity,

$$F(x)\to \text{const}\times x^{-7}(\nu\neq 0, x\to\infty). \qquad (22)$$

In the infrared limit, F has the universal behavior $F(x)=4x$ ($x\ll 1$), and thus we find a linear energy spectrum for $k\sqrt{t}\ll k_0\sqrt{t_0}$.

Chandrasekhar's self-similar solutions are easily generalized to flows in expanding Universes. The dependence on time and wave-number for the self-similar energy spectrum is [16]

$$E(k,t)=\frac{1}{2}v_t^2(t)\lambda(t)F(\lambda k) \qquad (23)$$

where λ and v_t are respectively Taylor's microscale and an average turbulent velocity, defined as

$$\lambda^2(t) \equiv 5\frac{\int E(k,t)dk}{\int E(k,t)k^2 dk}$$

$$\frac{1}{2}v_t^2(t) \equiv \int E(k,t)dk. \tag{24}$$

The second equation implies the normalization condition

$$\int_0^\infty F(x')dx' = 1. \tag{25}$$

To obtain a self-similar flow reducing to Eq. (19) in the flat space limit, we must require the time evolution laws:

$$\lambda^2(t) = \lambda_i^2 + 10\int_{t_i}^{t} \frac{\eta}{(p+\rho)a^2}dt$$

$$v_t = v_{ti}\left(\frac{(p+\rho)_i a_i^4}{(p+\rho)a^4}\right)\frac{\lambda_i}{\lambda(t)}. \tag{26}$$

The equation which determines the function $F(\lambda k)$ in Eq. (23) turns out to be

$$\int_0^x \left[F(x') - x'\frac{dF(x')}{dx'}\right]dx' = \left\{\frac{2}{5} + A\int_x^\infty \frac{\sqrt{F(x')}}{x'^{\,3/2}}dx'\right\}\int_0^x F(x')x'^{\,2}dx' \tag{27}$$

where A is a constant. This equation has the same structure as in flat space-time, Eq. (20), which means that assuming Heisenberg's hypothesis the spectrum is linear in k for length scales much bigger than the Taylor's microscale.

Observe that for flat space-time, the proportionality between the integral up to a certain wave number k of the inertia and the viscous forces is given by Eqs. (17) and (18). In the case of a FRW space-time, the autosimilar solution required by Tomita *et al.* (23) needs a time dependent ν_{curv} in Eq. (17), defined by the analog of Eq. (18) with A_{flat} replaced by $A_{curv} = 5A\eta a^2$.

D. Solving for the spectrum

Let us analyze in more detail the solutions of Eq. (27). We assume the normalization Eq. (25). By taking the $x\to\infty$ limit in Eq. (27) we find

$$\int_0^\infty F(x')x'^{\,2}dx' = 5. \tag{28}$$

Taking a derivative of Eq. (27) we get

$$1 - x\frac{F'}{F} = x^2\left\{\frac{2}{5} + A(G-H)\right\} \tag{29}$$

where

$$G = \int_x^\infty \frac{\sqrt{F(x')}}{x'^{\,3/2}}dx' \tag{30}$$

$$H = \frac{1}{\sqrt{Fx^7}}\int_0^x F(x')x'^{\,2}dx'. \tag{31}$$

Let us consider first the $x\to 0$ limit. Assume $F\propto x^\alpha$. The left hand side of Eq. (29) tends to a finite limit $1-\alpha$. On the right hand side, G and H behave as $x^{(\alpha-1)/2}$, so if $\alpha>0$, this side goes to zero. We must therefore have $\alpha=1$, and

$$F \sim Cx, \quad x\to 0 \tag{32}$$

where C is some constant.

In the $x\to\infty$ limit, assume again a power law behavior $F\propto x^{-\beta}$. Now $G\to 0$, so we must have $H\to 2/5A$. From Eq. (28) we know that in this limit $H\sim 5/\sqrt{Fx^7}$, so we must have $\beta=7$ and

$$F \sim \left(\frac{25A}{2}\right)^2 x^{-7}, \quad x\to\infty. \tag{33}$$

Taking into account both limiting behaviors and Eq. (25), the function F may be approximated as

$$F[x] = \frac{x}{[\alpha+\beta x^4]^2}, \quad \alpha = \left[\frac{25A\pi^2}{128}\right]^{1/3}, \quad \beta = \frac{2}{25A}. \tag{34}$$

III. EQUIVALENT FLUID FOR FIELD FLUCTUATIONS

After establishing the basic necessary notions for the description of hydrodynamic flows, our goal is to associate an equivalent fluid description to field fluctuations, and to derive the particle spectrum therefrom. Our first step is to obtain the energy density, pressure and velocity of this fluid as functionals of the quantum state of the field.

For simplicity, we shall consider the theory of a single, self-interacting scalar field ϕ, minimally coupled to gravity. The action is

$$S = \int d^4x\sqrt{-g}\left\{\left(\frac{-1}{2}\right)\partial_\mu\phi\partial^\mu\phi - V[\phi]\right\} \tag{35}$$

where $V[\phi]$ is a renormalized effective potential. The energy-momentum tensor is associated to the Heisenberg operator

$$T_Q^{\mu\nu} = \frac{(-2)}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left\{\left(\frac{1}{2}\right)\partial_\rho\phi\partial^\rho\phi + V[\phi]\right\}. \tag{36}$$

The macroscopic behavior of the field, however, may be described in terms of a c-number energy-momentum tensor

$$T^{\mu\nu}=T_C^{\mu\nu}+T_S^{\mu\nu} \tag{37}$$

where

$$T_C^{\mu\nu}=\langle T_Q^{\mu\nu}\rangle_Q \tag{38}$$

and $T_S^{\mu\nu}$ is a stochastic component with zero mean and self-correlation

$$\langle T_S^{\mu\nu}T_S^{\rho\sigma}\rangle_S=\frac{1}{2}\langle\{T_Q^{\mu\nu},T_Q^{\rho\sigma}\}\rangle_Q-T_C^{\mu\nu}T_C^{\rho\sigma}. \tag{39}$$

In these equations, $\langle\rangle_S$ denotes a stochastic average, while $\langle\rangle_Q$ is the average with respect to the quantum state of the field. Following Landau, we define the fluid four-velocity U^μ and energy density ρ as the (only) time-like eigenvector of $T^{\mu\nu}$ and (minus) its corresponding eigenvalue

$$T^{\mu\nu}U_\nu=-\rho U^\mu. \tag{40}$$

Introducing the pressure $p=p(\rho)$ as given by the equilibrium equation of state (our theory does not have a conserved particle number current, and therefore the equation of state is barotropic), we may decompose

$$T^{\mu\nu}=\rho U^\mu U^\nu+p\Delta^{\mu\nu}+\tau^{\mu\nu} \tag{41}$$

where $\Delta^{\mu\nu}=g^{\mu\nu}+U^\mu U^\nu$ and by construction $\tau^{\mu\nu}U_\nu=0$. Since $\tau^{\mu\nu}$ vanishes by definition in the equilibrium state, it may be parametrized in terms of deviations from equilibrium. Remaining within the so-called *first order formalism* [25,26], we may write

$$\tau^{\mu\nu}=-\eta H^{\mu\nu}-\zeta U^\rho_{,\rho}\Delta^{\mu\nu};\qquad \eta,\zeta\geqslant 0 \tag{42}$$

where

$$H^{\mu\nu}=\frac{1}{2}\Delta^{\mu\lambda}\Delta^{\nu\sigma}\left[U_{\lambda,\sigma}+U_{\sigma,\lambda}-\frac{2}{3}\Delta_{\lambda\sigma}U^\rho_{,\rho}\right] \tag{43}$$

and η and ζ are the *shear* and *bulk* viscosity coefficients, respectively.

Let us decompose each quantity in a mean component (denoted by a C subscript) and a fluctuation (denoted by an S). If the quantum state shares the symmetries of the FRW background, then $U_C^i=0$. Since $U^2=-1$ holds identically (as opposed to "in the mean") we must have

$$(U_C^0)^2+\langle(U_S^0)^2\rangle-a^2\langle U_S^iU_S^i\rangle_S=1 \tag{44}$$

$$2U_C^0U_S^0-a^2[U_S^iU_S^i-\langle U_S^iU_S^i\rangle_S]=0. \tag{45}$$

The second equation shows that U_S^0 is a higher order fluctuation with respect to U_S^i. If we remain at linear order, then we may approximate $U^0=U_C^0=1$. Observing that all mean values are homogeneous and isotropic, we see that τ^{0i} is also a higher order fluctuation. We find

$$T^{0i}=T_S^{0i}=(\rho+p)_C U_S^i \tag{46}$$

and therefore the velocity correlation

$$R^{ij}(\vec r,t)=\frac{a^2(t)}{2(\rho+p)_C^2(t)}\langle\{T_Q^{0i}(\vec r,t),T_Q^{0j}(0,t)\}\rangle_Q. \tag{47}$$

This is the key equation linking the quantum and stochastic descriptions. To estimate the velocity correlation, let us assume that, after integrating out the hard modes, the soft modes of interest may be described in terms of quasifree, long lived excitations with an effective mass $M^2(t)$. Then

$$T_Q^{0i}=\left(\frac{-1}{a^2(t)}\right)\partial_t\phi\,\partial_i\phi. \tag{48}$$

The fluctuations are Gaussian to a very good approximation, and therefore

$$\begin{aligned}&\langle\{T_Q^{0i}(\vec r,t),T_Q^{0j}(0,t)\}\rangle_Q\\&=\left(\frac{1}{a^4(t)}\right)\{\partial^2_{tj'}G^+\partial^2_{it'}G^++\partial^2_{ij'}G^+\partial^2_{tt'}G^+\\&+(\vec r\to-\vec r)\}\end{aligned} \tag{49}$$

where $G^+(x,x')$ is the positive frequency propagator

$$G^+((\vec r,t),(0,t))=\langle\phi(\vec r,t)\phi(0,t)\rangle_Q, \tag{50}$$

($\partial_{i,t}$ stand for derivatives with respect to the first argument, while $\partial_{i',t'}$ stand for derivatives with respect to the second argument of G^+). Let us decompose the soft field into modes

$$\phi(\vec r,t)=\int\frac{d^3k}{(2\pi)^3}e^{i\vec k\vec r}\phi_{\vec k}(t). \tag{51}$$

At any time t_0 we may introduce positive frequency adiabatic modes defined by [27]

$$f_k(t)=\frac{1}{\sqrt{2\omega_k(t)}}\exp\left\{-i\int_{t_0}^t dt'\,\omega_k(t')\right\} \tag{52}$$

where

$$\omega_k^2(t)=\frac{k^2}{a^2(t)}+M^2(t) \tag{53}$$

and decompose the mode amplitude $\phi_{\vec k}(t)$ into positive and negative frequency components

$$\phi_{\vec k}(t)=f_k(t)A_{\vec k}(t)+f_k^*(t)A^\dagger_{-\vec k}(t) \tag{54}$$

$$\partial_t\phi_{\vec k}(t)=-i\omega_k(t)\{f_k(t)A_{\vec k}(t)-f_k^*(t)A^\dagger_{-\vec k}(t)\}. \tag{55}$$

Let us define the spectrum

$$n_k(t)=\langle A^\dagger_{\vec k}(t)A_{\vec k}(t)\rangle_Q \tag{56}$$

(because the quantum state is isotropic, the spectrum depends only on k) and assume that

$$\langle A_{\vec{k}'}(t)A_{\vec{k}}(t)\rangle_Q=\langle A^\dagger_{\vec{k}'}(t)A^\dagger_{\vec{k}}(t)\rangle_Q$$
$$=\langle A^\dagger_{\vec{k}'}(t)A_{\vec{k}}(t)\rangle_Q|_{\vec{k}'\neq\vec{k}}=0 \qquad (57)$$

(this happens, for example, if the different modes acquire random phases through interaction with an environment). Then

$$G^+((\vec{r},t),(0,t'))$$
$$=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\vec{r}}\{f_k(t)f_k^*(t')[1+n_k(t)]$$
$$+f_k^*(t)f_k(t')n_k(t)\} \qquad (58)$$

and

$$\partial_tG^+((\vec{r},t),(0,t'))$$
$$=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\vec{r}}(-i\omega_k(t))\{f_k(t)f_k^*(t')[1+n_k(t)]$$
$$-f_k^*(t)f_k(t')n_k(t)\}. \qquad (59)$$

Observe that only the vacuum part contributes in the coincidence limit $t'\to t$. In the large occupation numbers regime we are interested in, this is negligible, and we get

$$\langle\{T_Q^{0i}(\vec{r},t),T_Q^{0j}(0,t)\}\rangle_Q$$
$$\sim\left(\frac{1}{a^4(t)}\right)\{\partial^2_{ij'}G^+\partial^2_{tt'}G^++(\vec{r}\to-\vec{r})\} \qquad (60)$$

where

$$\partial^2_{ij'}G^+=\frac{1}{3}\delta_{ij}\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\vec{r}}\left(\frac{k^2}{\omega_k(t)}\right)n_k(t) \qquad (61)$$

$$\partial^2_{tt'}G^+=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\vec{r}}\omega_k(t)n_k(t). \qquad (62)$$

We may now write down the Fourier transform of the velocity self correlation

$$\Phi^{ij}(k,t)=\frac{\delta^{ij}}{3a^2(t)(\rho+p)^2_C(t)}$$
$$\times\int\frac{d^3p}{(2\pi)^3}\frac{\omega_p(t)}{\omega_{|\vec{k}-\vec{p}|}(t)}|\vec{k}-\vec{p}|^2n_p(t)n_{|\vec{k}-\vec{p}|}(t). \qquad (63)$$

In principle, this is an integral equation relating the spectrum to the energy self-correlation. It may be simplified as follows. For small $p\leqslant k$, the integral reads

$$\left(\frac{k^2}{\omega_k(t)}\right)n_k(t)\int\frac{d^3p}{(2\pi)^3}\omega_p(t)n_p(t) \qquad (64)$$

while for large p we get

$$\int\frac{d^3p}{(2\pi)^3}p^2n_p^2(t). \qquad (65)$$

Since the spectra we are considering fall off much faster than p^{-2} (or even p^{-3}) at large p, we may estimate that the contribution from Eq. (64) is larger than Eq. (65); moreover, extending the integral to the whole momentum space,

$$\int\frac{d^3p}{(2\pi)^3}\omega_p(t)n_p(t)\sim\rho(t) \qquad (66)$$

and we get

$$\Phi^{ij}(k,t)=\frac{\rho(t)\delta^{ij}}{3a^2(t)(\rho+p)^2_C(t)}\left(\frac{k^2}{\omega_k(t)}\right)n_k(t). \qquad (67)$$

Finally, we may relate the particle spectrum to the turbulent energy spectrum

$$E(k,t)=\frac{2\pi\rho(t)}{a^2(t)(\rho+p)^2_C(t)}\left(\frac{k^4}{\omega_k(t)}\right)n_k(t). \qquad (68)$$

IV. REHEATING

In the previous sections, we have analyzed on one hand self-similar turbulent flows in expanding universes, and on the other hand have given the rule to translate the turbulent energy spectrum into a particle number distribution. We must now show that the foregoing analysis is relevant to plausible models of the reheating period, and use it to predict the likely shape of the final particle spectrum.

At this point, it is convenient to be more precise about the model of reheating we have in mind (for further details, see [1–5]). We assume that inflation is driven by an inflaton field Φ with an effective potential which may be parametrized as $V(\Phi)\sim\lambda_{inf}\Phi^4$. The self-coupling $\lambda_{inf}\sim10^{-14}$, and at the end of inflation $\Phi\sim m_p\sim10^{19}$ GeV (in natural units). This is the dominant contribution to the energy density, so it fixes the scale for the Hubble constant H:$m_p^2H^2\sim V(\Phi)$.

During preheating, a large fraction (nearly all) of this energy is transferred to the matter fields. These are described by an effective degree of freedom ϕ which self interacts with an effective dimensionless coupling constant g. For simplicity, we shall model this self interaction as a $g\phi^4$ theory. At times t_0 at the end of preheating, matter excitations are distributed with a smooth spectrum $n_k\sim Nf(k/k_0)$, where k is a comoving wave number and the characteristic momentum $k_0\geqslant H$ (meaning that the relevant modes are inside the horizon). We wish to find the form of the spectrum as follows from the assumption that its further evolution will be self-similar.

Let us estimate the energy density in the matter fields at t_0 as $\rho_0\sim Nk_0^4\sim\lambda_{inf}\Phi^4$. The self interaction of the matter fields induces a mass M^2. We shall assume $k_0^2\geqslant M^2$, in which case

$$M^2 \sim g \int \frac{d^3k}{k} n_k \sim gNk_0^2 \sim g(\rho_0/k_0^2). \tag{69}$$

We shall assume that this is the dominant contribution to the matter self-energy. The condition $k_0^2 \geqslant M^2$ is equivalent to

$$gN \leqslant 1 \tag{70}$$

(this equation will be important below). Observe that

$$\frac{M^2}{k_0^2} \sim \frac{g\rho_0}{k_0^4} \sim \left(\frac{g}{\lambda_{inf}}\right)\left(\frac{\rho_0}{k_0^4}\right)\left(\frac{V(\Phi)}{\Phi^4}\right)$$
$$\sim \left(\frac{g}{\lambda_{inf}}\right)\left(\frac{H}{k_0}\right)^4\left(\frac{m_p}{\Phi}\right)^4. \tag{71}$$

Since H/k_0 is already less than one, this is not unduly restrictive on g (see [1–5]). Also observe that M^2 redshifts with the cosmological evolution, so these estimates are not affected by Hubble flow.

Integrating Eq. (68) over k, we find the mean velocity in the equivalent fluid flow as

$$v_{ti}^2 \sim \frac{\rho_0 p_0}{(\rho_0 + p_0)^2} \leqslant c_s^2 \tag{72}$$

where

$$c_s^2 \sim \frac{k_{0phys}^2}{k_{0phys}^2 + M^2} \tag{73}$$

$[k_{0phys} = k_0/a(t)]$ is the speed of sound, so the flow may be regarded as incompressible. The shear viscosity of the matter fields may be estimated as (see the Appendix) [28]

$$\eta \sim \frac{k_0^3}{Ng^2} \tag{74}$$

leading to a kinematic viscosity

$$\nu \equiv \frac{\eta}{p+\rho} \sim \frac{1}{(gN)^2 k_0}. \tag{75}$$

With a typical velocity of order one, and a typical length of order k_0^{-1}, we get the Reynolds number $R \sim (gN)^2 \leqslant 1$ [cf. Eq. (70)].

Of course, we do not have a microscopic justification for Heisenberg's closure, so we shall take A in Eq. (27) as a free parameter. As it turns out, agreement with the numerical results reported by [1–5] is obtained for $A \sim 3$, which, as expected, corresponds to Reynolds numbers of order of one.

It is worth pointing out that Chandrasekhar's solution with the function F obeying Eq. (27) is an exact solution that relies only on Heisenberg's closure hypothesis Eq. (17). So if we trust this hypothesis, we can get the spectrum by solving Eq. (27) for any Reynolds number.

The distribution of occupation numbers is found from Eq. (68), where the energy spectrum is given by Eq. (23). In the light field limit $M^2 \ll k_0^2$, we find $\omega_k \sim k$. Then Eq. (32) implies $n_k \sim k^{-2}$ for $k \to 0$, and Eq. (33) implies k^{-10} for $k \to \infty$. This theoretical prediction for the exponents involved is the main result of this paper.

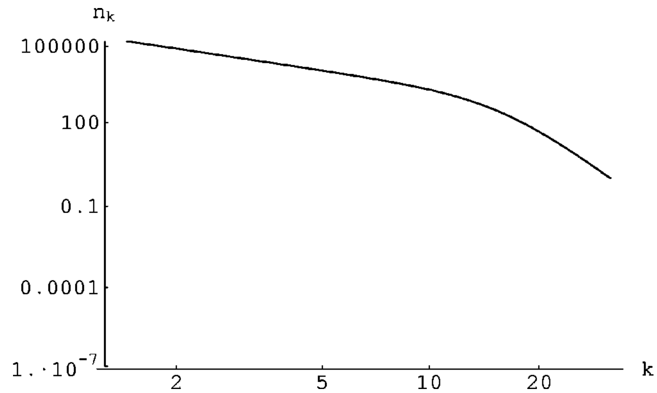

FIG. 1. Log-log plot of the particle spectrum, as given by Eq. (68). The energy spectrum is given by Eq. (23), where the function F is given by Eq. (34). We have chosen $A=3$, the Taylor microscale $\lambda \sim 0.18 \times 10^{-13}$ GeV^{-1}, and have scaled the spectrum to make it easiest to compare against the results presented in [3] and [4].

In Fig. 1 we show the full particle spectrum based on the approximation Eq. (34) for the function F. We have scaled the plot to make it easiest to compare with [1–5]. Momentum is measured in units of $a^{-1}10^{12}$ GeV (where a is the scale factor) [4], and we have chosen $\lambda \sim a1.8 \times 10^{-13}$ GeV^{-1}. Observing that in the ultrarelativistic limit the speed of flow $v_t \sim 1$ and $p \sim \rho$, integrating the particle density times k^3 shows that the total energy density in the flow is $\rho \sim 10^9 \times (10^{12}$ GeV$)^4$. The equivalent black-body temperature is then somewhat less than 10^{15} GeV, which is a reasonable value for the reheating era, and high enough to justify the neglect of all masses.

The analysis leading to the Chandrasekhar solutions begins from the Navier-Stokes equations for the fluid, which are equivalent in this context to energy-momentum conservation. Therefore our model requires that it be possible to assign to the fluid an independently conserved energy-momentum tensor. This is not exactly the same as requiring that the homogeneous mode has totally decayed, but it does mean that there is no significative energy transfer from the homogeneous mode to the fluid, either through parametric amplification or otherwise. The plots of total particle density and effective masses presented in Ref. [4] suggest that this condition is met early, then follows an intermediate stage dominated by rescattering, and finally the thermalization stage.

Comparison with the results in [1–5] is meaningful only in the intermediate phase. Decaying turbulence is necessarily a transient phenomenon. As time evolves, we expect the field will eventually thermalize, and the spectrum will get closer to a Rayleigh-Jeans tail, $n_k \sim k^{-1}$ when masses are negligible. The several plots presented in Ref. [4], where indexes go from 1.7 to slightly over 1, capture the transition from turbulence to equilibrium. Since the same plots show that earlier spectra are steeper (see also Fig. 1 in Ref. [3]) this is in satisfactory agreement with the prediction from self-similar flows.

V. FINAL REMARKS

In this paper, we have shown that the self-similar flows studied by Heisenberg, Chandrasekhar and Tomita may be used to provide an interpretation of the "turbulent "spectra found in [1–5]. The hydrodynamic model predicts scale invariant spectra $n_k \sim k^{-\alpha}$ both in the infrared and ultraviolet limits, with $\alpha \sim 2$ in the former, and 10 in the latter regime. Agreement with the early time results presented in [1–5] is satisfactory.

The connection of hydrodynamics to the behavior of fluctuations during reheating has interest of its own, as it provides an alternative to brute force quantum field theoretic calculations, and also yields physical insight on the macroscopic behavior of quantum fields in the early universe. The equivalent fluid method may be used to advantage also in other regimes, such as the inflationary period itself [29]. Moreover, it opens up a wealth of new phenomena, such as intermittence [14] and shocks [30], which are not apparent in the customary treatments. We will continue our research in this field, which promises a most rewarding dialogue between cosmology, astrophysics, and nonlinear physics at large.

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APPENDIX: ESTIMATE FOR THE MATTER VISCOSITY

In order to estimate the viscosity for the matter fields we assume that the correspondence Eq. (68) between the fluid and particle spectra allows us to associate a solution of the hydrodynamical evolution equations for the former to a self-similar solution of the kinetic equation for the latter (cf. [8]). We may then estimate the transport coefficients by adopting the same methods usually applied in equilibrium [31]. These are discussed in detail, in the quantum field theory context, in Ref. [28].

Let $n_k^0 \sim N f(X)$, $X = U^\mu k_\mu / k_0$ be a solution to the covariant Boltzmann equation

$$k^\mu \nabla_\mu n_k = I_{col}[n_k]. \tag{A1}$$

Since we are interested in the high energy regime where shear viscosity is much larger than bulk viscosity, we may use the Boltzmann collision integral

$$I_{col}[n_{k1}] \sim g^2 \int Dk_2 Dk_3 Dk_4 \delta(k_1 + k_2 - k_3 - k_4)\{(1+n_{k1}) \times (1+n_{k2}) n_{k3} n_{k4} - (1+n_{k3})(1+n_{k4}) n_{k1} n_{k2}\} \tag{A2}$$

where $Dk \sim d^4 k\, \delta(k^2)$. To estimate the shear viscosity we assume the four-velocity U^μ is slightly inhomogeneous, and seek a solution

$$n_k \sim n_k^0 + n_k^1 \tag{A3}$$

where the new term satisfies the equation (which we render only schematically)

$$\frac{N}{k_0} \frac{df}{dX} k^i k^j U_{i,j} \sim \frac{\delta I_{col}}{\delta n_k}[n_k^0] n_k^1. \tag{A4}$$

By simple power counting, we estimate

$$\frac{\delta I_{col}}{\delta n_k}[n_k^0] \sim g^2 N^2 k_0^2 \tag{A5}$$

and so

$$n_k^1 \sim \frac{1}{g^2 N k_0^3} \frac{df}{dX} k^i k^j U_{i,j}. \tag{A6}$$

The new term induces a correction to the energy-momentum tensor

$$T_{ij}^1 = \int Dk k_i k_j n_k^1. \tag{A7}$$

Equating this to $\eta U_{i,j}$ and repeating our power counting analysis, we obtain

$$\eta \sim \frac{k_0^3}{N g^2} \tag{A8}$$

as in Eq. (74). In equilibrium, $k_0 \to T$, $N \to 1$, and we obtain the same result as in Ref. [28].

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